Non-coherent Network Coding:
An Arbitrarily Varying Approach

Mahdi Jafari Siavoshani (EPFL)
Shenghao Yang (CUHK)
Raymond Yeung (CUHK)

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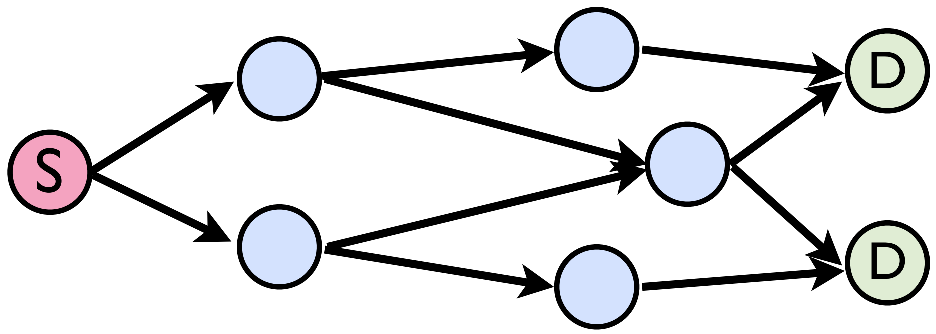
Outline

- Motivation and Introduction
- Problem Statement (Channel Model for NC NC)
- Partially Arbitrarily Varying Channel (PAVC)
- Non-coherent NC Capacity
- Conclusion

Motivation

- Consider a network performing randomized linear NC:
  - Nodes operation: linearly and randomly combine packets
    - Sources and destinations are oblivious to the relay nodes operations and network topology
    - There exist random delay, synchronization error, etc...
      - => a non-coherent transmission scenario



  - How do we model this transmission scenario?

Operator Channel - Subspace Coding

(Kotter and Kschischang 2008)

- Observation: The linear network coding is subspace preserving
    - Information transmission: is done by the injection of a basis for the vector space Π_S into the network
    - Information reception: is done by collecting a basis for the vector space Π_D by the receiver
- The network is modeled by the operator channel:

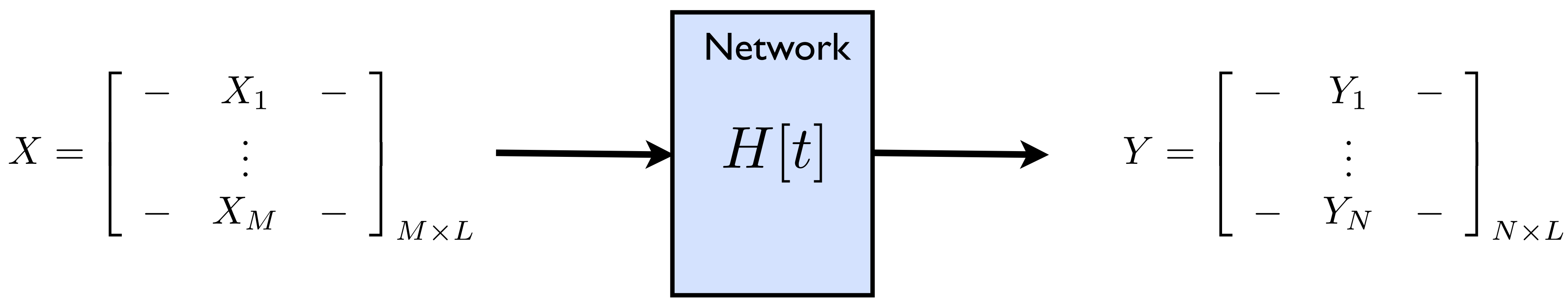
$$\Pi_D = \mathcal{H}_k(\Pi_S) \oplus \Pi_E$$

- [KK-IT08] focused on code construction in $\mathcal{P}(\mathbb{F}_q^L)$ which is a combinatorial problem
- They proposed subspace codes with block length one
- They didn't consider any specific probabilistic model for this channel

Matrix Channel

- The network operation can be modeled by a multiplicative matrix channel:

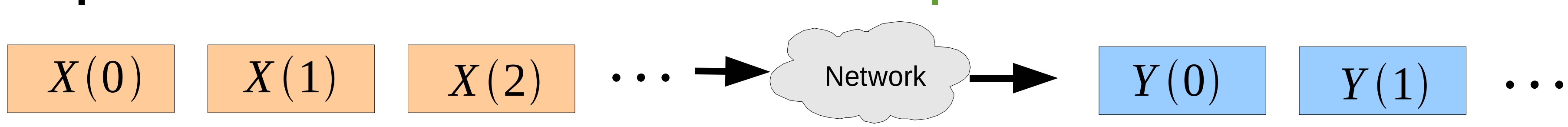
$$Y[t] = H[t]X[t]$$



- The input and output symbols are matrices over $\mathbb{F}_q$

- If we impose an appropriate probabilistic model => we can define capacity for this channel

- To achieve the capacity we might need a coding scheme over multiple blocks: a codeword is a sequence of matrices



Related Work

- Uniform and i.i.d. distribution over all matrices (Jafari et al):

$$C = i^*(L - i^*)\log q + o_q(1) \qquad i^* = \min[M, N, \lfloor L/2 \rfloor]$$

- Uniform and i.i.d. distribution over full-rank matrices (Silva et al):

$$C = \log\left(\sum_{i=0}^{M} \begin{bmatrix} L \\ i \end{bmatrix}_q\right)$$

- For a general distribution there are bounds (Yang et al):

$$(L - M)\mathbb{E}\left[\mathrm{rank}(H)\right]\log q + k \le C \le L\mathbb{E}\left[\mathrm{rank}(H)\right]\log q$$

- Uniform given rank distribution (u.g.r.) (Nobrega et al)

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- Uniform given rank distribution (u.g.r.) (Nobrega et al)
- Disadvantages of considering a specific distribution for $H[t]$:
  - The distribution of $H[t]$ depends on the network topology
  - It is very hard to relate transfer matrix distribution to the actual network topology => need for less restrictive model

Related Work

- R. Koetter and F. Kschischang, "Coding for errors and erasures in random network coding," IT, 2008.
- A. Montanari and R. Urbanke, "Coding for network coding," preprint.
- M. Jafari, S. Mohajer, C. Fragouli, and S. Diggavi, "On the Capacity of Noncoherent Network Coding," ISIT, 2009.
- D. Silva, F. R. Kschischang, and R. Koetter, "Communication over finite-field matrix channels," IT 2010.
- S. Yang, S.-W. Ho, J. Meng, and E.-h. Yang, "Optimality of subspace coding for linear operator channels over finite fields," ITW, 2010.
- S. Yang, S.-W. Ho, J. Meng, E. hui Yang, and R. W. Yeung, "On Linear operator channels over finite fields," preprint, 2010.
- R. W. Nobrega, B. F. Uchoa-Filho, D. Silva, "On the capacity of multiplicative finite-field matrix channels," ISIT, 2011.

Problem Setup: Alternative Model

- The non-coherent NC channel is modeled by a multiplicative matrix channel:

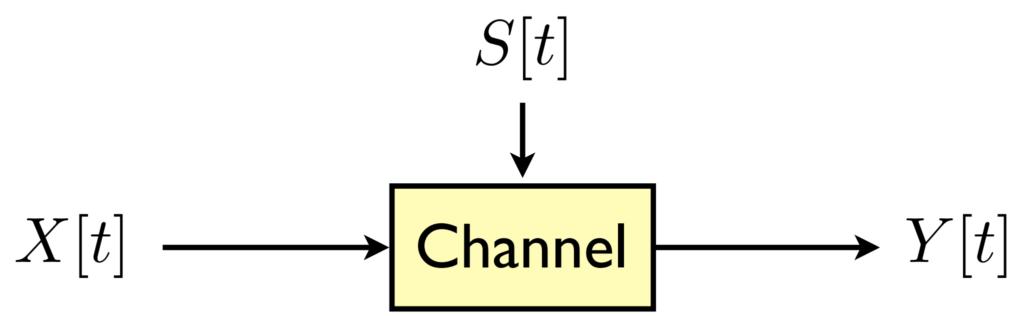
$$Y[t] = H[t]X[t]$$

- The channel transfer matrix is unknown to both Tx and Rx
- $H[t]$ changes arbitrarily from block to block with a constraint on its rank: the rank of $H[t]$ are i.i.d. and $H[t] \sim R$
- The distribution R is known by Tx and Rx

- This is similar to AVC model but with a probabilistic constraint
- This is a worst case model
- Advantage: even if we cannot find the rank distribution we can measure it in practice!

Partially Arbitrarily Varying Channel

- Arbitrarily varying channel (AVC):



$X \in \mathcal{X}$
$Y \in \mathcal{Y}$
$S \in \mathcal{S}$

- Channel state varies arbitrarily at every time-slot:

$$W^n(\boldsymbol{y}|\boldsymbol{x};\boldsymbol{s}) \triangleq \prod_{t=1}^{n} W(y_t|x_t;s_t)$$

- Partially arbitrarily varying channel (PAVC):
  - An AVC with a probabilistic constraint over the state space
  - Define a function $q : \mathcal{S} \to \mathcal{Q}$ where $\mathcal{Q} = \{0, \ldots, m\}$ then

$$P_{\boldsymbol{q}(\boldsymbol{S})}(q_1, \ldots, q_n) = \prod_{t=1}^{n} P_R(q_t)$$

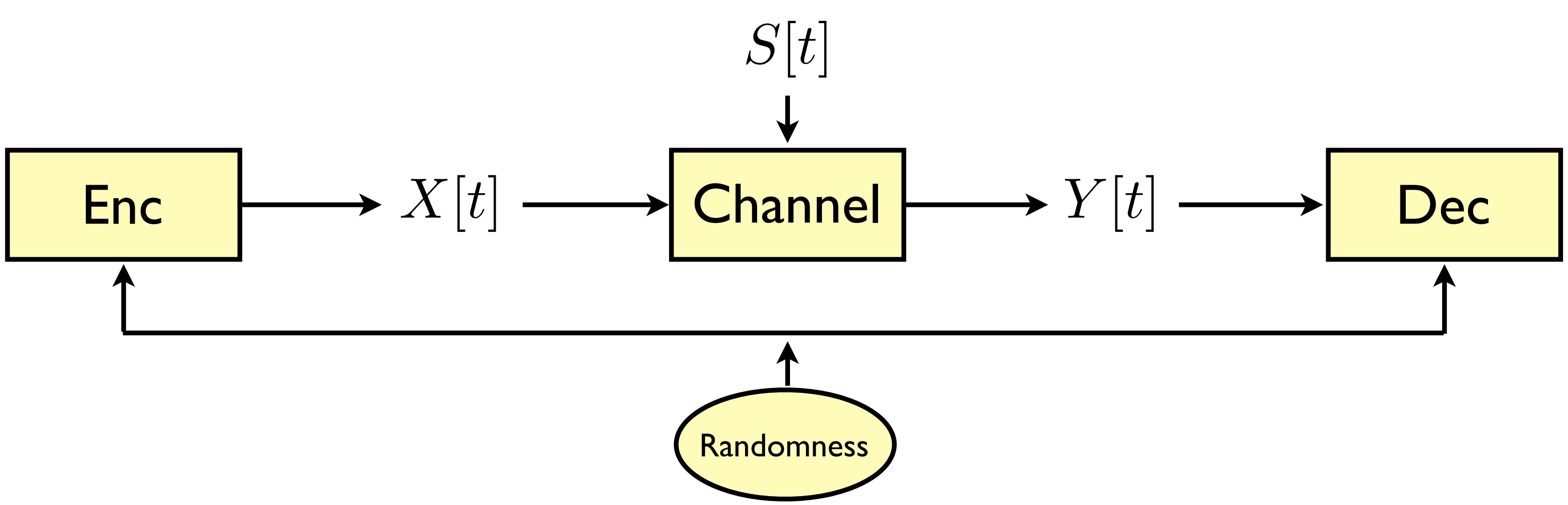
Partially Arbitrarily Varying Channel

- Similar to AVC, PAVC problem have many different variations
- For the average error probability, we have characterized the capacity of randomize and deterministic codes
- Theorem: The randomize code capacity of a PAVC is given by:

$$C_{\text{pavc}}^{\text{r,a}} = \max_{P_X} \min_{P_{S|q(S)}} I(P_X, \bar{W}_S) = \min_{P_{S|q(S)}} \max_{P_X} I(P_X, \bar{W}_S)$$

where

$$\bar{W}_S(y|x) \triangleq \mathbb{E}[W(y|x;S)]$$



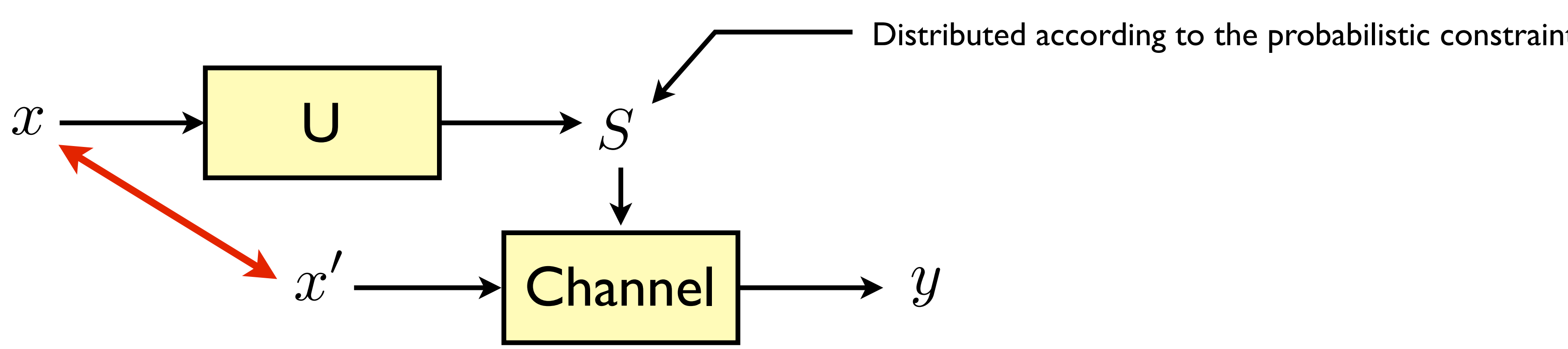
Partially Arbitrarily Varying Channel

- Theorem: The deterministic code capacity of a PAVC is non-zero iff the PAVC in non-symmetrizable. If $C_{\text{pavc}}^{\text{d,a}} > 0$ then we have

$$C_{\text{pavc}}^{\text{d,a}} = C_{\text{pavc}}^{\text{r,a}}$$

- Defenition: A PAVC is symmetrizable if for every x,x' and y we have:

$$\sum_s W(y|x;s)U(s|x',q(s))P_R(q(s)) = \sum_s W(y|x';s)U(s|x,q(s))P_R(q(s))$$



Capacity of Non-coherent NC

- The proposed model for the non-coherent NC is a PAVC
- Theorem: The randomized and deterministic code capacities of the proposed channel are the same and are equal to

$$C = \max_{P_X} \min_{P_{H|\mathrm{rk}(H)}} I(X;Y)$$

- For the deterministic code: we need to show that the channel is non-symmetrizable => it is hard to show this directly

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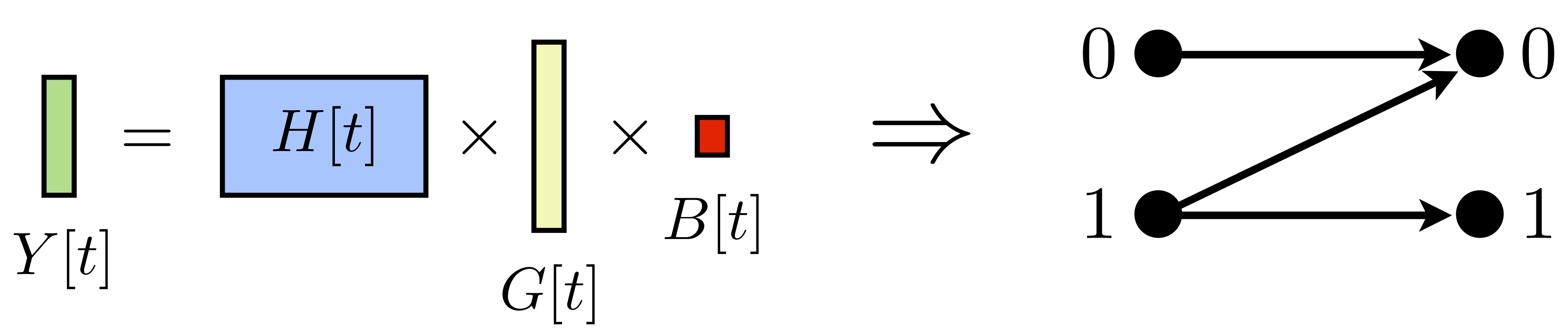

- So $C_{\mathrm{pavc}}^{\mathrm{d,a}} > 0$

Capacity of Non-coherent NC

- Lemma: The uniform given rank (u.g.r.) distribution for $P_{H|\mathrm{rk}(H)}$ is a minimizer in: $C = \max_{P_X} \min_{P_{H|\mathrm{rk}(H)}} I(X;Y)$

- Suppose that A and B are full-rank and uniformly at random chosen matrices => H and AHB have the same rank distribution but AHB is u.g.r.

- By data processing inequality: if $P^*_{H|\mathrm{rk}(H)}$ is a minimizer then the u.g.r. distribution with the same rank distribution is also a minimizer

Capacity of Non-coherent NC

- Theorem: $$C = \max_{P_X} I(X; \bar{H}X)$$

  where $\bar{H}$ has the same rank distribution as H but it is a u.g.r.

- [NUS-ISIT11]:
  - Corollary: Coding over subspaces is sufficient to achieve the capacity
  - The problem of finding capacity reduces to a convex optimization problem on $O(\min[M, L])$ variables

- Remark: A simpler proof is possible without the need for the PAVC results => for more discussion refer to the paper

Conclusion

- We have proposed a new model for the non-coherent network coding based on the notion of AVC
  - The NCNC has been modeled by a multiplicative matrix channel with arbitrarily varying transfer matrix which is subjected to a rank constraint
  - We have characterized the capacity for this model: it is equivalent to finding the capacity for a u.g.r. model
- To this end, we have extended the notion of AVC to an arbitrarily varying channel with probabilistic constraints on states
  - For the average probability of error, we have derived the capacity for the randomized and deterministic code

Thank You!